

# ON THE NUMBER OF FLATS TANGENT TO CONVEX HYPERSURFACES IN RANDOM POSITION

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**ABSTRACT.** Motivated by questions in real enumerative geometry [3, 4, 7, 8, 9, 10, 11], we investigate the problem of the number of flats simultaneously tangent to several *convex* hypersurfaces in real projective space from a probabilistic point of view. More precisely, we say that smooth convex hypersurfaces  $X_1, \dots, X_{d_{k,n}} \subset \mathbb{RP}^n$ , where  $d_{k,n} = (k+1)(n-k)$ , are in random position if each one of them is randomly translated by elements  $g_1, \dots, g_{d_{k,n}}$  sampled independently from the Orthogonal group with the uniform distribution; we denote by  $\tau_k(X_1, \dots, X_{d_{k,n}})$  the average number of  $k$ -dimensional projective subspaces (*flats*) which are simultaneously tangent to all the hypersurfaces. We prove that

$$\tau_k(X_1, \dots, X_{d_{k,n}}) = \delta_{k,n} \cdot \prod_{i=1}^{d_{k,n}} \frac{|\Omega_k(X_i)|}{|\text{Sch}(k, n)|},$$

where  $\delta_{k,n}$  is the *expected degree* from [4] (the average number of  $k$ -flats incident to  $d_{k,n}$  many random  $(n-k-1)$ -flats),  $|\text{Sch}(k, n)|$  is the volume of the Special Schubert variety of  $k$ -flats meeting a  $(n-k-1)$ -flat (computed in [4]) and  $|\Omega_k(X)|$  is the volume of the manifold of all  $k$ -flats tangent to  $X$ . We give a formula for the evaluation of  $|\Omega_k(X)|$  in term of some curvature integral of the embedding  $X \hookrightarrow \mathbb{RP}^n$  and we relate it with the classical notion of *intrinsic volumes* of a convex set:

$$\frac{|\Omega_k(\partial C)|}{|\text{Sch}(k, n)|} = 4|V_{n-k-1}(C)|, \quad k = 0, \dots, n-1.$$

As a consequence we prove that

$$\tau_k(X_1, \dots, X_{d_{k,n}}) \leq \delta_{k,n} \cdot 4^{d_{k,n}}$$

## 1. INTRODUCTION

**1.1. Flats simultaneously tangent to several hypersurfaces.** Given  $X_1, \dots, X_{d_{k,n}} \subset \mathbb{RP}^n$  hypersurfaces, with  $d_{k,n} = (k+1)(n-k)$ , a classical problem in enumerative geometry is to determine how many  $k$ -dimensional projective subspaces (called *flats*) of  $\mathbb{RP}^n$  are simultaneously tangent to  $X_1, \dots, X_{d_{k,n}}$ .

Geometrically we can formulate this problem as follows. Let  $\mathbb{G}(k, n)$  denote the Grassmannian of  $k$ -dimensional projective subspaces of  $\mathbb{RP}^n$  (note that  $d_{k,n}$  equals the dimension of  $\mathbb{G}(k, n)$ ). If  $X \subset \mathbb{RP}^n$  is a smooth hypersurface, we denote by  $\Omega_k(X) \subset \mathbb{G}(k, n)$  the variety of  $k$ -tangents to  $X$ , i.e. the set of  $k$ -flats that are tangent to  $X$  at some point. The number of  $k$ -flats simultaneously tangent to  $X_1, \dots, X_{d_{k,n}}$  equals

$$\# \Omega_k(X_1) \cap \dots \cap \Omega_k(X_{d_{k,n}}).$$

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Of course this number depends on the mutual position of  $X_1, \dots, X_{d_{k,n}}$  in projective space.

In [9] F. Sottile and T. Theobald considered the problem of tangents to spheres in affine space  $\mathbb{R}^n$ . They showed that for  $2n - 2$  general spheres in  $\mathbb{R}^n$  there are (at most)  $3 \cdot 2^{n-1}$  complex (real) affine lines tangent to all of them and found a configuration of spheres where all these lines are real. (In this case we speak of a *fully real* problem.) They also studied [10] the problem of  $k$ -flats tangent to  $d_{k,n}$  many general quadrics in  $\mathbb{RP}^n$  and proved that the “complex bound”  $2^{d_{k,n}} \cdot \deg(\mathbb{G}_{\mathbb{C}}(k, n))$  can be attained by real quadrics (i.e. the problem is fully real). See also [3, 7, 8, 11] for other interesting results on real enumerative geometry of tangents.

Motivated by this, an exciting point of view comes by adopting a probabilistic approach, asking instead for the *expected* answer. Specifically one can assume that the hypersurfaces  $X_1, \dots, X_{d_{k,n}}$  are in *random position* in space, meaning that each one of them is randomly translated by elements  $g_1, \dots, g_{d_{k,n}}$  sampled independently from the Orthogonal group with the uniform distribution. The average number of  $k$ -flats tangent to the hypersurfaces  $X_1, \dots, X_{d_{k,n}}$  in random position is then given by

$$\tau_k(X_1, \dots, X_{d_{k,n}}) := \mathbb{E} \# \Omega_k(X_1) \cap \dots \cap \Omega_k(X_{d_{k,n}}).$$

The computation and study of properties of this number, in the case when the hypersurfaces are boundaries of convex sets, is precisely the goal of this paper – we note a novel feature of our work: we leave the semialgebraic setting and move to the convex framework.

**Definition 1.1** (Convex hypersurface). *A subset  $C$  of  $\mathbb{RP}^n$  is called strictly convex if  $C$  does not intersect some hyperplane  $L$  and it is strictly convex in the affine space  $\mathbb{RP}^n \setminus L$ . We will call  $X \subset \mathbb{RP}^n$  a convex hypersurface whenever it bounds a smooth, strictly convex open set.*

*Remark 1.2* (Spherical versus projective geometry). Our considerations in projective spaces run parallel to what happens on spheres, with just small adaptations. A set  $C \subset S^n$  is called strictly convex if it is the intersection of a convex cone  $K \subset \mathbb{R}^{n+1}$  with  $S^{n-1}$ . For the purposes of enumerative geometry, the notion of *flats* should be replaced with the one of *great spheres* (linear sections of  $S^n$ ). Computations involving volumes and the generalized integral geometry formula also require very small modifications (mostly multiplications by a factor of two) and we leave them to the reader.

*Example 1.3* (Two circles in the plane). Consider the real projective plane  $\mathbb{RP}^2$  with the standard Riemannian metric (the one which makes the double covering  $S^2 \rightarrow \mathbb{RP}^2$  a local isometry) and let  $X_1 = X_2$  be two copies of a circle of radius  $r$  in  $\mathbb{RP}^2$  (i.e. each  $X_i$  consists of the set of points at distance  $r$ , in the projective plane, from a given point  $x_i$ , and it is isometric to the boundary of a spherical cap of radius  $r$  in  $S^2$ ). In this case

$$\begin{aligned} \tau_0(X_1, X_2) &= \mathbb{E} \# \{\text{points in } X_1 \cap gX_2\} \\ &= \frac{|X_1|}{|\mathbb{RP}^1|} \cdot \frac{|X_2|}{|\mathbb{RP}^1|} \\ &= \frac{2\pi \sin r}{\pi} \cdot \frac{2\pi \sin r}{\pi} = (2 \sin r)^2 \end{aligned}$$

(we have used integral geometry to evaluate the expectation in the first step, since  $g \in O(3)$  is uniformly distributed). The case of lines tangent to both  $X_1$  and  $X_2$  is also elementary and can be treated as follows. There are 2 lines tangent to both if the center  $gx_2$  of the circle  $gX_2$  is inside the disk  $D(x_1, 2r)$  (the case when  $X_1$  and  $gX_2$  coincide has probability zero); the number

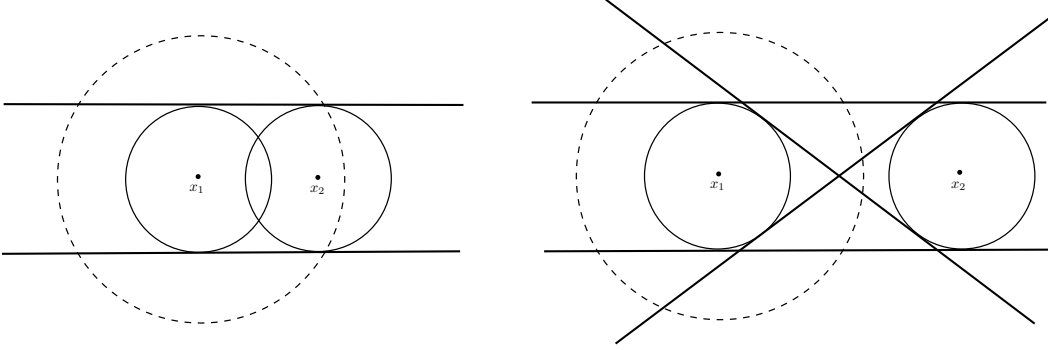


FIGURE 1. Arrangements of circles of the same radius  $r$  in the (projective) plane. If the centers are less than twice the radius apart there are only 2 lines tangent to both, otherwise the number of lines is 4. When the center are uniformly distributed on  $\mathbb{RP}^2$ , there are on average  $(2 \cos r)^2$  many lines tangent to both.

of lines tangent to both is 4 if  $x_2$  is outside  $D(x_1, 2r)$  (see Figure 1). Hence:

$$\begin{aligned} \tau_1(X_1, X_2) &= 2 \cdot \mathbb{P}\{x_2 \in D(x_1, 2r)\} + 4 \cdot (1 - \mathbb{P}\{x_2 \in D(x_1, 2r)\}) \\ &= 4 - 2 \cdot \frac{|D(x_1, 2r)|}{|\mathbb{RP}^2|} \\ &= 4 - 2 \cdot \frac{2\pi(1 - \cos 2r)}{2\pi} = (2 \cos r)^2. \end{aligned}$$

**1.2. Probabilistic enumerative geometry.** Recently, the second author of the current paper together with P. Bürgisser, have studied the similar problem of determining the average number of  $k$ -flats that simultaneously intersect  $d_{k,n}$  many  $(n - k - 1)$ -flats in random position in  $\mathbb{RP}^n$ . They have called this number the *expected degree* of the real Grassmannian  $\mathbb{G}(k, n)$ , here denoted by  $\delta_{k,n}$ , and have claimed that this is the key quantity governing questions in the random enumerative geometry of flats. (The name comes from the fact that the number of solutions of the analogous problem over the complex numbers coincides with the degree of  $\mathbb{G}_{\mathbb{C}}(k, n)$  in the Plücker embedding.)

For reasons that will become more clear later, it is convenient to introduce the *special Schubert variety*<sup>1</sup>  $\text{Sch}(k, n) \subset \mathbb{G}(k, n)$  consisting  $k$ -flats in  $\mathbb{RP}^n$  intersecting a fixed  $(n - k - 1)$ -flat. The volume of the special Schubert variety is computed in [4, Theorem 4.2]:

$$|\text{Sch}(k, n)| = |\mathbb{G}(k, n)| \cdot \frac{\Gamma\left(\frac{k+2}{2}\right) \Gamma\left(\frac{n-k+1}{2}\right)}{\Gamma\left(\frac{k+1}{2}\right) \Gamma\left(\frac{n-k}{2}\right)},$$

where  $|\mathbb{G}(k, n)|$  denotes the volume of the Grassmannian. Next theorem relates our main problem to the expected degree (see Theorem 4.1 below).

**Theorem** (Probabilistic enumerative geometry). *The average number of  $k$ -flats in  $\mathbb{RP}^n$  simultaneously tangent to convex hypersurfaces  $X_1, \dots, X_{d_{k,n}}$  in random position equals*

$$\tau_k(X_1, \dots, X_{d_{k,n}}) = \delta_{k,n} \cdot \prod_{i=1}^{d_{k,n}} \frac{|\Omega_k(X_i)|}{|\text{Sch}(k, n)|},$$

<sup>1</sup>Note that in the notation of [4] we have  $\text{Sch}(k, n) = \Sigma(k + 1, n + 1)$  and  $\delta_{k,n} = \text{edeg } G(k + 1, n + 1)$ .

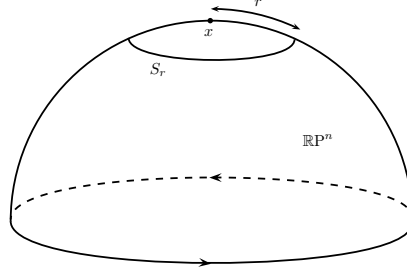


FIGURE 2. The equation  $x_1^2 + \dots + x_n^2 = (\tan r)^2 x_0^2$  defines in  $\mathbb{RP}^n$  a sphere of radius  $r$  (i.e. the set of all points at distance  $r$  from a given point).

where  $|\Omega_k(X)|$  denotes the volume of the manifold of  $k$ -tangents.

The number  $\delta_{k,n}$  equals (up to a multiple) the volume of a convex body for which the authors of [4] coined the name *Segre zonoid*. Except for  $\delta_{0,n} = \delta_{n-1,n} = 1$ , the exact value of this quantity is not known, but it is possible to compute its asymptotic as  $n \rightarrow \infty$  for fixed  $k$ . For example, in the case of the Grassmannian of lines in  $\mathbb{RP}^n$  one has [4, Theorem 6.8]

$$(1.1) \quad \delta_{1,n} = \frac{8}{3\pi^{5/2}} \cdot \frac{1}{\sqrt{n}} \cdot \left(\frac{\pi^2}{4}\right)^n \cdot (1 + \mathcal{O}(n^{-1})).$$

The number  $\delta_{1,3}$  (the average number of lines meeting four random lines in  $\mathbb{RP}^3$ ) can be written as an integral [4, Proposition 6.7], whose numerical approximation is  $\delta_{1,3} = 1.7262\dots$ . It is an open problem whether this quantity has a closed formula (possibly in terms of special functions).

This reduces our study to the investigation of the geometry of the manifold of tangents, for which we prove the following result (Propositions 3.1 and 3.2 below).

**Proposition** (The volume of the manifold of  $k$ -tangents). *For a convex hypersurface  $X \subset \mathbb{RP}^n$  we have*

$$(1.2) \quad \frac{|\Omega_k(X)|}{|\text{Sch}(k, n)|} = \frac{\Gamma\left(\frac{k+1}{2}\right) \Gamma\left(\frac{n-k}{2}\right)}{\pi^{\frac{n+1}{2}}} \int_X \sigma_k(x) dV_X.$$

where  $\sigma_k : X \rightarrow \mathbb{R}$  is the  $k$ -th elementary symmetric polynomial of the principal curvatures of the embedding  $X \hookrightarrow \mathbb{RP}^n$ .

*Example 1.4* (Spheres in projective space). Let us consider the case of spheres of equal radius  $0 < r < \frac{\pi}{2}$  in  $\mathbb{RP}^3$ , i.e. for  $i = 1, \dots, d_{k,3}$  and  $k = 0, 1, 2$  we take  $X_i = \{x_1^2 + x_2^2 + x_3^2 = (\tan r)^2 x_0^2\}$  (see Figure 2). Then, applying the above results, we get

$$\tau_0(X_1, X_2, X_3) = \delta_{0,3} \cdot (\sqrt{2} \sin r)^6 \quad (\delta_{0,3} = 1)$$

$$\tau_1(X_1, X_2, X_3, X_4) = \delta_{1,3} \cdot \left(\frac{8}{\pi} \sin r \cos r\right)^4 \quad (\delta_{1,3} = 1.72\dots)$$

$$T_2(X_1, X_2, X_3) = \delta_{2,3} \cdot (\sqrt{2} \cos r)^6 \quad (\delta_{2,3} = 1).$$

More generally, when  $X_i = S_{r_i} = \{x_1^2 + \dots + x_n^2 = (\tan r_i)^2 x_0^2\} \subset \mathbb{RP}^n$  is a  $(n-1)$ -sphere of radius  $r_i \in (0, \frac{\pi}{2})$  in  $\mathbb{RP}^n$ , all its principal curvatures are constants equal to  $\cot r_i$  and Corollary 3.2 gives

$$\frac{|\Omega_k(S_r)|}{|\text{Sch}(k, n)|} = \frac{2\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{k+2}{2}\right) \Gamma\left(\frac{n-k+1}{2}\right)} \cdot (\cos r_i)^k (\sin r_i)^{n-k-1},$$

since  $|S_{r_i}| = \frac{2\sqrt{\pi}^n}{\Gamma(\frac{n}{2})}(\sin r_i)^{n-1}$ . Combining this into Theorem 4.1 we obtain

$$(1.3) \quad \tau_k(S_{r_1}, \dots, S_{r_{d_{k,n}}}) = \delta_{k,n} \cdot \prod_{i=1}^{d_{k,n}} \left( \frac{2\Gamma(\frac{n+1}{2})}{\Gamma(\frac{k+2}{2})\Gamma(\frac{n-k+1}{2})} \cdot (\cos r_i)^k (\sin r_i)^{n-k-1} \right).$$

(Note that, as before, the cases of  $\tau_0$  and  $\tau_{n-1}$  can also be treated with “elementary” geometric considerations.) It is interesting to study, for a fixed  $k$ , the maximum that the expectation can be in the case when all the hypersurfaces are spheres. In the case  $k = 1$  we see that the quantity  $\cos r_i (\sin r_i)^{n-2}$  is maximized when  $r = \arccos \frac{1}{\sqrt{n-1}}$  (note that this is  $\frac{\pi}{2} - \frac{1}{n^{1/2}} + O(n^{-1/2})$ , just a bit smaller than  $\frac{\pi}{2}$ ) and

$$\begin{aligned} \max_{r \in (0, \pi/2)} \frac{|\Omega_k(S_r)|}{|\text{Sch}(k, n)|} &= \frac{4}{\sqrt{\pi}} \cdot \frac{\left(\frac{n-2}{n-1}\right)^{\frac{n-2}{2}}}{(n-1)^{\frac{1}{2}}} \cdot \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})} \\ &= \left(\frac{8}{e\pi}\right)^{\frac{1}{2}} \left(1 + \frac{1}{2n} + O(n^{-2})\right). \end{aligned}$$

In particular, using the asymptotic (1.2) and (1.4) we obtain

$$\begin{aligned} \max_{r_1, \dots, r_{2n-2} \in (0, \frac{\pi}{2})} \tau_1(S_{r_1}, \dots, S_{r_{2n-2}}) &= \delta_{1,n} \cdot \left( \left(\frac{8}{e\pi}\right)^{\frac{1}{2}} \left(1 + \frac{1}{2n} + O(n^{-2})\right) \right)^{2n-2} \\ (1.4) \quad &= \frac{e^2}{3\pi^{\frac{3}{2}}} \cdot \frac{1}{\sqrt{n}} \cdot \left(\frac{2\pi}{e}\right)^n \cdot (1 + O(n^{-1})). \end{aligned}$$

We observe that a hypersurface which is a sphere in an affine chart  $U \simeq \mathbb{R}^n$  is a convex hypersurface in the projective space  $\mathbb{RP}^n$ , but it is not a sphere (in our terminology) unless it’s centered at the origin (and viceversa a sphere in projective space needs not be a sphere in an affine chart). In fact (1.4) tells that Sottile and Theobald’s upper bound  $3 \cdot 2^{n-1}$  for the number of lines tangent to  $d_{1,n}$  affine spheres in  $\mathbb{R}^n$  *does not* apply to the case of *projective* spheres: since  $\frac{2\pi}{e} > 2$ , when  $n$  is large (1.4) is larger than  $3 \cdot 2^{n-1}$ ; as a consequence there must be a configuration of  $d_{1,n}$  projective spheres in  $\mathbb{RP}^n$  with (exponentially) more commonly tangent lines.

**1.3. A new interpretation of intrinsic volumes.** The quantities  $|\Omega_k(X)|$  offer a new interesting interpretation of the classical notion of *intrinsic volumes*. Recall that if  $C$  is a convex set in  $\mathbb{RP}^n$ , the spherical Steiner’s formula allows to write the volume of an  $\epsilon$ -neighborhood  $C_\epsilon$  of  $C$  in  $\mathbb{RP}^n$  as

$$|C_\epsilon| = |C| + \sum_{k=0}^{n-1} f_{n,k}(\epsilon) V_k(C),$$

where the functions  $f_{n,k}$  do not depend on  $C$  and are defined by

$$(1.5) \quad f_{n,k}(\epsilon) = \frac{4\pi^{\frac{n+1}{2}}}{\Gamma(\frac{k+1}{2})\Gamma(\frac{n-k}{2})} \int_0^\epsilon (\cos t)^k (\sin t)^{n-1-k} dt.$$

The quantities  $V_0(C), \dots, V_{n-1}(C)$  are called intrinsic volumes of  $C$ . What is remarkable is that when  $C$  is smooth and strictly convex,  $|\Omega_k(\partial C)|$  coincides, up to a constant depending on  $k$  and  $n$  only, with the  $(n-k-1)$ -th intrinsic volume of  $C$ .

**Proposition** (The manifold of  $k$ -tangents and intrinsic volumes). *Let  $C \subset \mathbb{RP}^n$  be a smooth strictly convex set, then:*

$$|V_{n-k-1}(C)| = \frac{1}{4} \cdot \frac{|\Omega_k(\partial C)|}{|\text{Sch}(k, n)|}, \quad k = 0, \dots, n-1.$$

This interpretation allows to derive a nice surprising property (Corollary 3.4), which offers possible new directions of investigation and allows to prove the upper bound<sup>2</sup> (Corollary 4.2)

$$\tau_k(X_1, \dots, X_{d_{k,n}}) \leq \delta_{k,n} \cdot 4^{d_{k,n}}.$$

**1.4. Related work.** Enumerative geometry over the field of complex numbers is classical. Over the Reals it is a much harder subject, due to the nonexistence of generic configurations. From the deterministic point of view we mention, among others, the papers that are closest to our work, and that gave a motivation for it: [3, 7, 8, 9, 10, 11]. The probabilistic approach to real enumerative geometry was initiated in [4] for what concerns Schubert calculus, and in [2] for the study of the number of real lines on random hypersurfaces.

## 2. PRELIMINARIES

By  $\mathbb{G}(k, n) \simeq Gr(k+1, n+1)$  we denote the grassmanian of  $(k+1)$ -planes in  $\mathbb{R}^{n+1}$  (or, equivalently, the set of projective  $k$ -flats in  $\mathbb{RP}^n$ ). Both notations will be used throughout the article. The dimension of  $\mathbb{G}(k, n)$  is denoted by  $d_{k,n} := \dim(\mathbb{G}(k, n)) = (k+1)(n-k)$ .

**2.1. Metrics & volumes.** The grassmanian  $Gr(k, n)$  is endowed with an  $O(n)$ -invariant riemannian metric through the Plücker embedding

$$i : Gr(k, n) \hookrightarrow P\left(\bigwedge^k \mathbb{R}^n\right)$$

where  $P(\bigwedge^k \mathbb{R}^n)$  denotes the projectivization of the vector space  $\bigwedge^k \mathbb{R}^n$  and is endowed with the standard metric. Using this we will locally identify  $Gr(k, n)$  with unit simple  $k$ -vectors  $v_1 \wedge \dots \wedge v_k$ , where  $v_1, \dots, v_k$  are orthonormal in  $\mathbb{R}^n$ .

A canonical left-invariant metric on the orthogonal group  $O(n)$  is defined as

$$\langle A, B \rangle := \frac{1}{2} \text{tr}(A^t B), \quad A, B \in T_1 O(n)$$

Denoting by  $|X|$  the total volume of a riemannian manifold  $X$  whenever it is finite one can show the following formulas

$$|Gr(k, n)| = \frac{|O(n)|}{|O(k)||O(n-k)|}, \quad \frac{|O(n+1)|}{|O(n)|} = |S^n|, \quad |O(1)| = 2, \quad |S^n| = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})}.$$

For a semialgebraic subset  $X$  of  $Gr(k, n)$  we denote by  $|X|$  the volume of its smooth locus.

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<sup>2</sup>The following upper bound [4, Theorem 6.5] is asymptotically sharp (in the logarithmic scale)

$$\delta_{k,n} \leq \frac{|\mathbb{G}(k, n)|}{|\mathbb{RP}^{d_{k,n}}|} \left( \sqrt{\frac{2\pi}{k+1}} \cdot \frac{\Gamma\left(\frac{k+2}{2}\right)}{\Gamma\left(\frac{k+1}{2}\right)} \right)^{d_{k,n}}$$

**2.2. Probabilistic setup.** Given a riemannian manifold  $Y$  and a smooth function  $f : Y \rightarrow \mathbb{R}$  we denote by  $\int_Y f(y) dV_Y$  the integration of  $f$  with respect to the riemannian volume density of  $Y$ . We recall that there is a unique  $O(n)$ -invariant probability distribution on  $O(n), Gr(k, n)$  and  $S^n$  called *uniform* (see [4, 6] for more details). For a measurable subset  $A \subset X \in \{O(n), Gr(k, n), S^n\}$  it is defined as

$$\mathbb{P}(A) := \frac{1}{|X|} \int_X \mathbf{1}_A dV_X.$$

In the sequel all probabilistic concepts will be referred to the above listed spaces endowed with the uniform distribution.

*Remark 2.1.* For a measurable  $A \subset Gr(k, n)$  the set  $\hat{A} = \{g \in O(n) : g^{-1}\mathbb{R}^k \in A\}$  is measurable in  $O(n)$  and

$$\mathbb{P}(A) = \frac{1}{|Gr(k, n)|} \int_{Gr(k, n)} \mathbf{1}_A dV_{Gr(k, n)} = \frac{1}{|O(n)|} \int_{O(n)} \mathbf{1}_{\hat{A}} dV_{O(n)} = \mathbb{P}(\hat{A})$$

We will implicitly use this identification when needed.

**2.3. Integral geometry of coisotropic hypersurfaces of grassmanian.** A smooth (or semi-algebraic) hypersurface  $\mathcal{H}$  of  $\mathbb{G}(k, n)$  is said *coisotropic* if for all (smooth codimension 1 points)  $\Lambda \in \mathcal{H}$  the normal space  $N_\Lambda \mathcal{H} \subset T_\Lambda \mathbb{G}(k, n) \simeq Hom(\Lambda, \Lambda^\perp)$  is spanned by a rank one operator.

For  $k, m \geq 1$  let  $u_j \in S(\mathbb{R}^k), v_j \in S(\mathbb{R}^m), j = 1, \dots, km$  be unit independent random vectors. Then the *average scaling factor*  $\alpha(k, m)$  is defined as

$$\alpha(k, m) := \mathbb{E} \|(u_1 \otimes v_1) \wedge \dots \wedge (u_{km} \otimes v_{km})\|$$

where  $\|\cdot\|$  is induced from the standart scalar product on  $\mathbb{R}^k \otimes \mathbb{R}^m$ :  $(u_1 \otimes v_1, u_2 \otimes v_2) := (u_1, u_2)(v_1, v_2)$ . We will use the generalized Poincaré formula for coisotropic hypersurfaces of  $\mathbb{G}(k, n)$  proved in [4, Thm. 3.19]:

**Theorem 2.2.** *Let  $\mathcal{H}_1, \dots, \mathcal{H}_{d_{k,n}}$  be coisotropic hypersurfaces of  $\mathbb{G}(k, n)$ . Then*

$$\mathbb{E} \#(g_1 \mathcal{H}_1 \cap \dots \cap g_{d_{k,n}} \mathcal{H}_{d_{k,n}}) = \alpha(k+1, n-k) |\mathbb{G}(k, n)| \prod_{i=1}^{d_{k,n}} \frac{|\mathcal{H}_i|}{|\mathbb{G}(k, n)|}$$

where  $g_1, \dots, g_{d_{k,n}} \in O(n+1)$  are independent randomly chosen orthogonal transformations.

*Remark 2.3.* This theorem expresses the average number of points in the intersection of  $d_{k,n}$  many hypersurfaces of  $\mathbb{G}(k, n)$  in *random position* in terms of the volumes of the hypersurfaces and the constant  $\alpha(k+1, n-k)$ , which only depends on the pair  $(k, n)$ .

**2.4. Intersection of special real Schubert varieties.** A *special real Schubert variety*  $Sch(k, n)$  consists of all projective  $k$ -flats in  $\mathbb{RP}^n$  that intersect a fixed projective  $(n-k-1)$ -flat  $\Pi$ :

$$Sch(k, n) = \{\Lambda \in \mathbb{G}(k, n) : \Lambda \cap \Pi \neq \emptyset\}$$

It is a coisotropic algebraic hypersurface of  $\mathbb{G}(k, n)$ . In [4] P. Bürgisser and the second author of the current article had introduced a notion of *expected degree*  $\delta_{k,n}$  of the grassmanian  $\mathbb{G}(k, n)$ . It is defined as the average number of projective  $k$ -flats in  $\mathbb{RP}^n$  simultaneously intersecting  $d_{k,n}$  many random projective  $(n-k-1)$ -flats independently chosen in  $\mathbb{G}(n-k-1, n)$ . In other words,

$$\delta_{k,n} := \mathbb{E} \#(g_1 Sch(k, n) \cap \dots \cap g_{d_{k,n}} Sch(k, n)).$$

Using the formula in [4, Thm. 4.2] for the volume of  $\text{Sch}(k, n)$ :

$$|\text{Sch}(k, n)| = |\mathbb{G}(k, n)| \frac{\Gamma(\frac{k+2}{2})}{\Gamma(\frac{k+1}{2})} \frac{\Gamma(\frac{n-k+1}{2})}{\Gamma(\frac{n-k}{2})}$$

and Theorem 2.2 one can express

$$\delta_{k,n} = \alpha(k+1, n-k) |\mathbb{G}(k, n)| \left( \frac{\Gamma(\frac{k+2}{2})}{\Gamma(\frac{k+1}{2})} \frac{\Gamma(\frac{n-k+1}{2})}{\Gamma(\frac{n-k}{2})} \right)^{d_{k,n}}$$

*Remark 2.4.* The exact value of  $\delta_{k,n}$  (equivalently  $\alpha(k+1, n-k)$ ) remains unknown for  $0 < k < (n-1)$ . See [4, Sect. 6] for various asymptotics of  $\delta_{k,n}$ .

*Remark 2.5.* Note that one can define a notion of “expected degree” even over the complex numbers, by sampling complex projective subspaces uniformly from the complex Grassmannian. Denoting by  $c_{k,n} \in H^2(\mathbb{G}_{\mathbb{C}}(k, n); \mathbb{Z})$  the first Chern class of the tautological bundle and by  $[\mathbb{G}_{\mathbb{C}}(k, n)] \in H_{2d_{k,n}}(\mathbb{G}_{\mathbb{C}}(k, n); \mathbb{Z})$  the orientation class, then

$$\text{expected degree over the complex numbers} = \langle (c_{k,n})^{d_{k,n}}, [\mathbb{G}_{\mathbb{C}}(k, n)] \rangle$$

The resulting number also equals the degree of  $\mathbb{G}_{\mathbb{C}}(k, n)$  in the Plücker embedding.

### 3. THE MANIFOLD OF TANGENTS

Let  $X = \partial A$  be a convex hypersurface of  $\mathbb{RP}^n$  (corresponding to a strictly convex open set  $A \subset \mathbb{RP}^n$ ) and let  $p : Gr_k(X) \rightarrow X$  be the grassmanian bundle of  $k$ -flats of  $X$ : (recall that this is a smooth fiber bundle whose fiber  $p^{-1}(x)$  is the grassmanian  $Gr_k(T_x X) \simeq Gr(k, n-1)$ ). Define a map

$$\begin{aligned} \psi : Gr_k(X) &\rightarrow \mathbb{G}(k, n) \\ (x, \Lambda) &\mapsto P(\text{span}\{x, \Lambda\}) \end{aligned}$$

where we abuse notation and we think at  $\Lambda \subset T_x X$  as a pair  $(x, \Lambda)$  and identify the tangent space  $T_x \mathbb{RP}^n$  with a hyperplane  $x^\perp \subset \mathbb{R}^{n+1}$  and thus  $\Lambda$  and  $x$  (a line in  $\mathbb{R}^{n+1}$ ) are both subspaces of  $\mathbb{R}^{n+1}$ .

With this notation we observe that  $\psi$  is a smooth embedding and that  $\Omega_k(X)$ , the set of all  $k$ -flats tangent to  $X$ , coincides by definition with  $\text{im}(\psi)$ .

Let's choose a unit normal vector field  $\nu$  to  $X \subset \mathbb{RP}^n$  pointing inside the convex region  $A$ . Then the second fundamental form  $B$  of  $X$  is positive definite everywhere. For  $(x, \Lambda) \in Gr_k(X)$  and an orthonormal basis  $v_1, \dots, v_k$  of  $\Lambda$  let's denote by  $B_x(\Lambda) = \det(B(v_i, v_j))$  the determinant of the  $k \times k$  matrix  $\{B(v_i, v_j)\}$ . Note that  $B_x(\Lambda)$  does not depend on the choice of  $v_1, \dots, v_k$ . Using the smooth coarea formula we prove the following

**Proposition 3.1.** *If  $X \subset \mathbb{RP}^n$  is a convex hypersurface, then*

$$(3.1) \quad |\Omega_k(X)| = \frac{|Gr(k, n-1)|}{\binom{n-1}{k}} \int_X \sigma_k(x) dV_X$$

where  $\sigma_k : X \rightarrow \mathbb{R}$  is the  $k$ -th elementary symmetric polynomial of the principal curvatures of the embedding  $X \hookrightarrow \mathbb{RP}^n$ .



*Proof.* The  $O(n+1)$ -invariant metric  $g$  on  $\mathbb{G}(k, n)$  induces a riemannian metric  $\psi^*g$  on  $Gr_k(X)$  through the embedding  $\psi$ . Note that the restriction of  $\psi^*g$  to the fibers  $Gr_k(T_x X)$  is  $O(T_x X) \simeq O(n-1)$ -invariant. Now we apply the smooth coarea formula to  $p : (Gr_k(X), \psi^*g) \rightarrow (X, g_X)$ , where  $g_X$  is a restriction to  $X$  of the invariant metric on  $\mathbb{RP}^n$ :

$$|\Omega_k(X)| = \int_{Gr_k(X)} dV_{Gr_k(X)} = \int_X \int_{Gr_k(T_x X)} (NJ_{(x, \Lambda)} p)^{-1} dV_{Gr_k(T_x X)} dV_X.$$

Let's show first that the normal Jacobian  $NJ_{(x, \Lambda)} p$  equals  $|B_x(\Lambda)|^{-1} = |\det(B(v_i, v_j))|^{-1}$ .

Given a point  $x \in X$ , a unit normal  $\nu \in T_x \mathbb{RP}^n$  to  $T_x X$  and an orthonormal basis  $v_1, \dots, v_k \in T_x X$  of  $\Lambda \in Gr_k(T_x X)$  let's complete them to an orthonormal basis  $x, \nu, v_1, \dots, v_k, v_{k+1}, \dots, v_{n-1}$  of  $\mathbb{R}^{n+1}$ . Using these vectors we describe the tangent space to  $Gr_k(X)$  at  $(x, \Lambda)$ .

For  $i = 1, \dots, n-1$  and  $j = 1, \dots, k$  let  $x_i = x_i(t)$  be a small curve through  $x$  in the direction  $v_i$  and let  $v_j^i = v_j^i(t)$  be the parallel transport of  $v_j$  along  $x_i$ , i.e. the vector field solving  $\nabla_{\dot{x}_i}^X v_j^i = 0, v_j^i(0) = v_j$ . Note that for any time  $t$  the vectors  $v_1^i(t), \dots, v_k^i(t) \in T_{x_i(t)} X$  remain pairwise orthonormal. Consider now curves in  $Gr_k(X)$  and their tangents produced by these vectors:

$$\begin{aligned} \tilde{\gamma}_i(t) &= (x_i(t), v_1^i(t) \wedge \dots \wedge v_k^i(t)) \\ \tilde{\Gamma}_i &:= \dot{\tilde{\gamma}}_i(0) = (v_i, \sum_{j=1}^k v_1 \wedge \dots \wedge \dot{v}_j^i(0) \wedge \dots \wedge v_k) \end{aligned}$$

Observe that

$$\dot{v}_j^i(0) = \nabla_{v_i}^{\mathbb{R}^{n+1}} v_j^i = \underbrace{\nabla_{v_i}^X v_j^i}_{=0} + a_{ij} x + b_{ij} \nu = a_{ij} x + b_{ij} \nu$$

Since the standard scalar product on  $\mathbb{R}^{n+1}$  (here denoted by a dot) induces the metric on  $T_x \mathbb{RP}^n = T_x S^n = x^\perp$  and since the second fundamental form of the unit sphere  $S^n \subset \mathbb{R}^{n+1}$  coincides with the metric we have

$$\begin{aligned} a_{ij} &= (\nabla_{v_i}^{\mathbb{R}^{n+1}} v_j^i) \cdot x = \delta_{ij} \\ (3.2) \quad b_{ij} &= (\nabla_{v_i}^{\mathbb{R}^{n+1}} v_j^i) \cdot \nu = (\nabla_{v_i}^{\mathbb{RP}^n} v_j^i + \delta_{ij} x) \cdot \nu = (\nabla_{v_i}^{\mathbb{RP}^n} v_j^i) \cdot \nu = B(v_i, v_j) \end{aligned}$$

The tangent space to the fiber  $T_{(x, \Lambda)} Gr_k(T_x X) = \text{Ker } p_*$  is spanned by the following  $k(n-1-k)$  vectors

$$\begin{aligned} \tilde{\theta}_{ij}(t) &= (x, v_1 \wedge \dots \wedge (v_i \cos t + v_j \sin t) \wedge \dots \wedge v_k), \quad i = 1, \dots, k \\ \tilde{\Theta}_{ij} &:= \dot{\tilde{\theta}}_{ij}(0) = (0, v_1 \wedge \dots \wedge v_j \wedge \dots \wedge v_k), \quad j = k+1, \dots, n-1 \end{aligned}$$

We will work with the images  $\Gamma_i, \Theta_{ij} \in T_{Span\{x, \Lambda\}} \mathbb{G}(k, n)$  of  $\tilde{\Gamma}_i$  and  $\tilde{\Theta}_{ij}$  under  $\psi_*$ . It is easy to see that

$$\begin{aligned} \Gamma_i &= \psi_* \tilde{\Gamma}_i = v_i \wedge v_1 \wedge \dots \wedge v_k + \sum_{j=1}^k b_{ij} x \wedge v_1 \wedge \dots \wedge v_j \wedge \dots \wedge v_k, \quad 1 \leq i \leq n-1 \\ \Theta_{ij} &= \psi_* \tilde{\Theta}_{ij} = x \wedge v_1 \wedge \dots \wedge v_j \wedge \dots \wedge v_k, \quad 1 \leq i \leq k, \quad k+1 \leq j \leq n-1 \end{aligned}$$

and  $\Gamma_i$ 's are orthogonal to  $\Theta_{ij}$ 's, but  $\Gamma_i$ 's are not in general orthonormal vectors. Therefore, since  $p_* \tilde{\Gamma}_i = v_i$  and the  $v_i$ 's form an orthonormal basis for  $T_x X$  in order to compute the normal Jacobian  $NJ_{(x, \Lambda)} p$  we need to find a change of basis matrix from  $\{\Gamma_i\}_{1 \leq i \leq n-1}$  to some

orthonormal basis of  $\text{Span}\{\Gamma_i\}_{1 \leq i \leq n-1} = \text{Ker}(p_* \circ \psi_*^{-1})^\perp$ . For this purpose let's note that for the orthonormal vectors

$$\begin{aligned} S_j &= x \wedge v_1 \wedge \cdots \wedge \underset{j}{\nu} \wedge \cdots \wedge v_k, & 1 \leq j \leq k \\ P_i &= v_i \wedge v_1 \wedge \cdots \wedge v_k, & k+1 \leq i \leq n-1 \end{aligned}$$

we have

$$\begin{pmatrix} \Gamma_1 \\ \vdots \\ \Gamma_k \\ \Gamma_{k+1} \\ \vdots \\ \Gamma_{n-1} \end{pmatrix} = \begin{pmatrix} b & 0 \\ * & 1 \end{pmatrix} \begin{pmatrix} S \\ R \end{pmatrix} = \begin{pmatrix} b_{11} & \cdots & b_{1k} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{k1} & \cdots & b_{kk} & 0 & 0 & \cdots & 0 \\ b_{k+1,1} & \cdots & b_{k+1,k} & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{n-1,1} & \cdots & b_{n-1,k} & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} S_1 \\ \vdots \\ S_k \\ P_{k+1} \\ \vdots \\ P_{n-1} \end{pmatrix}$$

where  $b = \{b_{ij}\}_{1 \leq i, j \leq k} = \{B(v_i, v_j)\}_{1 \leq i, j \leq k}$  by (3). Since  $\psi_*$  is injective the  $\Gamma_i$ 's are independent and hence  $b$  is invertible. Then

$$\begin{pmatrix} S_1 \\ \vdots \\ S_k \\ P_{k+1} \\ \vdots \\ P_{n-1} \end{pmatrix} = \begin{pmatrix} b^{-1} & 0 \\ * & 1 \end{pmatrix} \begin{pmatrix} \Gamma_1 \\ \vdots \\ \Gamma_k \\ \Gamma_{k+1} \\ \vdots \\ \Gamma_{n-1} \end{pmatrix}$$

Applying  $p_* \circ \psi_*^{-1}$  to the  $S_j, P_i$ 's we obtain that

$$NJ_{(x, \Lambda)} p = |\det(b^{-1})| = |B_x(\Lambda)|^{-1}$$

and thus

$$(3.3) \quad |\Omega_k(X)| = \int_X \int_{Gr_k(T_x X)} |B_x(\Lambda)| dV_{Gr_k(T_x X)} dV_X.$$

Since the fibers  $Gr_k(T_x X)$  are endowed with  $O(n-1) \simeq O(T_x X) \simeq O(\{x, \nu_x\}^\perp)$ -invariant metric we may rewrite the inner integral as

$$(3.4) \quad \int_{Gr_k(T_x X)} |B_x(\Lambda)| dV_{Gr_k(T_x X)} = |Gr(k, n-1)| \mathbb{E}_{\Lambda \in Gr(k, n-1)} |B_x(\Lambda)|$$

Since the restriction  $B|_\Lambda$  of a positive definite form  $B$  is also positive definite, we have  $B_x(\Lambda) > 0$  and

$$\mathbb{E}_{\Lambda \in Gr(k, n-1)} |B_x(\Lambda)| = \mathbb{E}_{\Lambda \in Gr(k, n-1)} B_x(\Lambda).$$

Now we prove that

$$\mathbb{E}_{\Lambda \in Gr(k, n-1)} B_x(\Lambda) = \binom{n-1}{k}^{-1} s_k(d_1(x), \dots, d_{n-1}(x))$$

where the  $d_i(x)$ 's are principal curvatures at the point  $x$  of  $X \subset \mathbb{RP}^n$  and  $s_k$  is the  $k$ -th elementary symmetric polynomial. Now let's choose an o.n.b.  $e = \{\delta_1, \dots, \delta_{n-1}\}$  of  $T_x X$  in which the second fundamental form  $B$  is diagonal  $D = \text{diag}\{d_1, \dots, d_{n-1}\}$ . For vectors  $v_i$  we denote by the same

letters their coordinate representation in the basis  $e$ . Let  $V$  and  $E$  be  $(n-1) \times k$  matrices with columns  $\{v_i\}_{1 \leq i \leq k}$  and  $\{\delta_i\}_{1 \leq i \leq k}$  respectively:

$$V = \begin{pmatrix} | & & | \\ v_1 & \dots & v_k \\ | & & | \end{pmatrix} \quad E = \begin{pmatrix} 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 1 \\ 0 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & 0 \end{pmatrix}$$

There exists an orthogonal matrix  $g \in O(n-1)$  s.t.  $V = g \cdot E$  and then  $b = \{B(v_i, v_j)\}_{1 \leq i, j \leq k}$  can be written as  $b = V^t D V = E^t g^t D g E$ . In this view  $B_x(\Lambda) = \det b = \det(E^t g^t D g E)$  is just the leading principal minor of  $g^t D g$  of order  $k$ . Note that  $B_x(\Lambda)$  does not depend on the choice of  $g$ , namely it's invariant under the action of  $\text{Stab}_{\text{Span}\{\delta_1, \dots, \delta_k\}} \simeq O(k) \times O(n-1-k) \subset O(n-1)$ . Using this and the fact that the induced metric on the fibers  $\text{Gr}_k(T_x X) \simeq \text{Gr}(k, n-1)$  is the standard  $O(n-1)$ -invariant metric we obtain

$$\begin{aligned} \mathbb{E}_{\Lambda \in \text{Gr}(k, n-1)} B_x(\Lambda) &= \frac{1}{|\text{Gr}(k, n-1)|} \int_{\text{Gr}(k, n-1)} B_x(\Lambda) dV_{\text{Gr}(k, n-1)} \\ &= \frac{1}{|\text{Gr}(k, n-1)| \cdot |O(k)| \cdot |O(n-1-k)|} \int_{O(n-1)} \det(E^t g^t D g E) dg \\ &= \frac{1}{|O(n-1)|} \int_{O(n-1)} \det(E^t g^t D g E) dg \end{aligned}$$

where  $dg = dV_{O(n-1)}$  is the invariant Haar measure on  $O(n-1)$ .

Now for any  $k$ -subset  $I = \{i_1, \dots, i_k\} \subset \{1, \dots, n-1\}$  denote  $E_I$  the  $(n-1) \times k$  matrix with columns  $\delta_{i_1}, \dots, \delta_{i_k}$ .  $E_I$  can be obtained as a left multiplication of  $E$  by a permutation matrix  $M_{\sigma_I}$ :  $E_I = M_{\sigma_I} \cdot E$ , where  $\sigma_I$  sends  $1, \dots, k$  into  $i_1, \dots, i_k$  respectively. Using invariance of  $dg$  we get

$$\int_{O(n-1)} \det(E_I^t g^t D g E_I) dg = \int_{O(n-1)} \det(E^t (g M_{\sigma_I})^t D (g M_{\sigma_I}) E) dg = \int_{O(n-1)} \det(E^t g^t D g E) dg$$

Consequently we can express  $\mathbb{E}_{\Lambda \in \text{Gr}(k, n-1)} B_x(\Lambda)$  as a sum over all  $k$ -subsets  $I \subset \{1, \dots, n-1\}$  divided by  $\binom{n-1}{k}$ :

$$\mathbb{E}_{\Lambda \in \text{Gr}(k, n-1)} B_x(\Lambda) = \binom{n-1}{k}^{-1} \frac{1}{|O(n-1)|} \int_{O(n-1)} \sum_{\substack{I \subset \{1, \dots, n-1\}, \\ |I|=k}} \det(E_I^t g^t D g E_I) dg$$

The integrand here is the sum of all principal minors of  $g^t D g$  of order  $k$  and thus does not depend on  $g$  and is equal to the  $k$ -th elementary symmetric polynomial  $s_k(d_1, \dots, d_{n-1})$  of  $\{d_1, \dots, d_{n-1}\}$ . Combining this with (3) and (3) we end the proof.  $\square$

In particular we can derive the following

**Corollary 3.2.** *If  $X \subset \mathbb{RP}^n$  is the boundary of a smooth strictly convex set  $C$ , then*

$$(3.5) \quad \frac{|\Omega_k(X)|}{|\text{Sch}(k, n)|} = \frac{\Gamma\left(\frac{k+1}{2}\right) \Gamma\left(\frac{n-k}{2}\right)}{\pi^{\frac{n+1}{2}}} \int_X \sigma_k(x) dV_X.$$

*Proof.* We first observe that

$$\frac{|Gr(k, n-1)|}{|\mathbb{G}(k, n)|} = \frac{1}{\pi^{n/2}} \frac{\Gamma(\frac{n}{2}) \Gamma(\frac{n+1}{2})}{\Gamma(\frac{k+1}{2}) \Gamma(\frac{n-k}{2})}$$

and, recalling [4, Theorem 4.2],

$$\frac{|\text{Sch}(k, n)|}{|\mathbb{G}(k, n)|} = \frac{|\Sigma(k+1, n+1)|}{|G(k+1, n+1)|} = \frac{\Gamma(\frac{k+2}{2})}{\Gamma(\frac{k+1}{2})} \cdot \frac{\Gamma(\frac{n-k+1}{2})}{\Gamma(\frac{n-k}{2})}.$$

Substituting into (3.1) we obtain

$$\begin{aligned} \frac{|\Omega_k(X)|}{|\text{Sch}(k, n)|} &= \frac{|Gr(k, n-1)|}{|\mathbb{G}(k, n)|} \cdot \frac{|\mathbb{G}(k, n)|}{|\text{Sch}(k, n)|} \cdot \frac{1}{\binom{n-1}{k}} \int_X \sigma_k(x) dV_X \\ &= \frac{1}{\pi^{n/2}} \frac{\Gamma(\frac{n}{2}) \Gamma(\frac{n+1}{2})}{\Gamma(\frac{k+1}{2}) \Gamma(\frac{n-k}{2})} \cdot \frac{\Gamma(\frac{k+1}{2}) \Gamma(\frac{n-k}{2})}{\Gamma(\frac{k+2}{2}) \Gamma(\frac{n-k+1}{2})} \cdot \frac{1}{\binom{n-1}{k}} \int_X \sigma_k(x) dV_X \\ &= \frac{1}{\pi^{n/2}} \frac{\Gamma(\frac{n}{2}) \Gamma(\frac{n+1}{2})}{\Gamma(\frac{k+2}{2}) \Gamma(\frac{n-k+1}{2})} \cdot \frac{1}{\binom{n-1}{k}} \int_X \sigma_k(x) dV_X \\ &= \frac{\Gamma(\frac{k+1}{2}) \Gamma(\frac{n-k}{2})}{\pi^{\frac{n+1}{2}}} \int_X \sigma_k(x) dV_X. \end{aligned}$$

□

**3.1. Intrinsic volumes.** Recall that when  $C \subset \mathbb{RP}^n$  is a convex sets, the intrinsic volumes  $V_0(C), \dots, V_{n-1}(C)$  are characterized by Steiner's formula, which gives the exact expansion (for small  $\epsilon$ ) of the volume of the  $\epsilon$ -neighborhood:

$$(3.6) \quad |C_\epsilon| = |C| + \sum_{k=0}^{n-1} f_{n,k}(\epsilon) V_k(C)$$

(the functions  $f_{n,k}$  are defined in (1.3)). Note that (3.1) in the projective case is obtained simply by looking at the geometry of the riemannian submersion  $p : S^n \rightarrow \mathbb{RP}^n$ : an open hemisphere in  $S^n$  is isometric to  $\mathbb{RP}^n$  minus a hyperplane. In fact one can consider the convex set  $\tilde{C} \subset S^n$  (one of the two pieces of  $p^{-1}(C)$ ) and apply the spherical version of Steiner's formula [5] (see also [1, Proposition 4.4.1]) for the computation of  $|\tilde{C}_\epsilon|$  which, for small values of  $\epsilon > 0$ , equals  $|C_\epsilon|$ . As a consequence we obtain the following.

**Corollary 3.3** (The manifold of  $k$ -tangents and intrinsic volumes). *Let  $C \subset \mathbb{RP}^n$  be a smooth strictly convex set, then:*

$$|V_{n-k-1}(C)| = \frac{1}{4} \cdot \frac{|\Omega_k(\partial C)|}{|\text{Sch}(k, n)|}, \quad k = 0, \dots, n-1.$$

*Proof.* We write Spherical Steiner's formula in two different ways. First [1, Proposition 4.4.1] reads:

$$(3.7) \quad |C_\epsilon| = |C| + \sum_{k=0}^{n-1} |T_{S^n}(S^k, \epsilon)| \cdot V_k(C),$$

where  $T_{S^n}(S^k, \epsilon)$  denotes the  $\epsilon$ -neighborhood of an equator  $S^k$  in  $S^n$ . On the other hand, in the smooth case [5] (see also [1, Corollary 4.3.3]) we have:

$$(3.8) \quad |C_\epsilon| = |C| + \sum_{k=0}^{n-1} |T_{S^n}(S^k, \epsilon)| |S^k| |S^{n-k-1}| \int_{\partial C} \sigma_{n-k-1}(x) dV_{\partial C}.$$

Since the function  $f_{n,0}, \dots, f_{n,n-1}$  are independent, comparing (3.1) with (3.1) we obtain

$$\begin{aligned} V_{n-k-1}(C) &= \frac{1}{|S^k||S^{n-k-1}|} \int_{\partial C} \sigma_k(x) dV_{\partial C} \\ &= \frac{\Gamma\left(\frac{k+1}{2}\right) \Gamma\left(\frac{n-k}{2}\right)}{4\pi^{\frac{n+1}{2}}} \int_X \sigma_k(x) dV_{\partial C} \\ &\stackrel{(3.2)}{=} \frac{1}{4} \cdot \frac{|\Omega_k(\partial C)|}{|\text{Sch}(k, n)|}. \end{aligned}$$

□

From this we derive the following interesting corollary.

**Corollary 3.4.** *Let  $X \subset \mathbb{RP}^n$  be the boundary of a smooth strictly convex set  $C$  and denote by  $C^\circ$  the polar of  $\tilde{C} \subset S^n$ . Then:*

$$\frac{4|C|}{|S^n|} + \frac{4|C^\circ|}{|S^n|} + \sum_{k=0}^{n-1} \frac{|\Omega_k(X)|}{|\text{Sch}(k, n)|} = 4.$$

In particular for every  $k = 0, \dots, n-1$  we have

$$(3.9) \quad \frac{|\Omega_k(X)|}{|\text{Sch}(k, n)|} \leq 4.$$

*Proof.* This follows immediately from the property of intrinsic volumes, see for instance [1, Proposition 4.4.10]. □

#### 4. HYPERSURFACES IN RANDOM POSITION

**Theorem 4.1.** *The average number of  $k$ -planes in  $\mathbb{RP}^n$  simultaneously tangent to convex hypersurfaces  $X_1, \dots, X_{d_{k,n}}$  in random position equals*

$$(4.1) \quad \tau_k(X_1, \dots, X_{d_{k,n}}) = \delta_{k,n} \cdot \prod_{i=1}^{d_{k,n}} \frac{|\Omega_k(X_i)|}{|\text{Sch}(k, n)|}.$$

*Proof.* We use the generalized kinematic formula for coisotropic hypersurfaces of  $\mathbb{G}(k, n)$  proved in [4] (Theorem 2.2 above).

In order to apply Theorem 2.2 to the case  $\mathcal{H}_i = \Omega_k(X_i), i = 1, \dots, d_{k,n}$ , we need to prove that each  $\Omega_k(X_i)$  is a coisotropic hypersurface of  $\mathbb{G}(k, n)$ . For this purpose fix  $(x, \Lambda) \in Gr_k(X)$  s.t.  $\psi((x, \Lambda)) = P(\text{span}\{x, \Lambda\}) \in \Omega_k(X_i)$ . As in the proof of Proposition 3.1 let's consider an orthonormal basis  $v_1, \dots, v_{n-1}$  of  $T_x X$  s.t.  $\Lambda = \text{span}\{v_1, \dots, v_k\}$  and a unit normal  $\nu \in T_x \mathbb{RP}^n$  to  $T_x X$ . For a curve  $x_\nu(t) \subset \mathbb{RP}^n$  through  $x$  in the direction  $\nu$  we consider parallel transports  $v_1^\nu(t), \dots, v_k^\nu(t) \in T_{x_\nu(t)} \mathbb{RP}^n$  of  $v_1, \dots, v_k$  along  $x_\nu(t)$ . We claim that the tangent to the curve  $\gamma(t) = x_\nu(t) \wedge v_1^\nu(t) \wedge \dots \wedge v_k^\nu(t) \in \mathbb{G}(k, n)$  is normal to  $T_{x \wedge v_1 \wedge \dots \wedge v_k} \Omega_k(X_i)$ . Indeed,

$$\dot{\gamma}(0) = \nu \wedge v_1 \wedge \dots \wedge v_k + \sum_{j=1}^k x \wedge v_1 \wedge \dots \wedge \dot{v}_j^\nu(0) \wedge \dots \wedge v_k = \nu \wedge v_1 \wedge \dots \wedge v_k$$

since  $\dot{v}_j^\nu(0) = \nabla_\nu^{\mathbb{RP}^n} v_j^\nu + a_j x = 0 + a_j x$  is proportional to  $x$ . In (3.1) we already found the tangent space to  $\Omega_k(X_i)$  and it elementary to verify that that  $\dot{\gamma}(0)$  is orthogonal to it. Seen as an operator  $\dot{\gamma}(0)$  sends  $x$  in  $\nu$  and all vectors in  $\Lambda$  in 0. Hence  $\Omega_k(X_i)$  is coisotropic.

Applying now Theorem 2.2 we deduce

$$(4.2) \quad \tau_k(X_1, \dots, X_{d_{k,n}}) = \alpha(k+1, n-k) |\mathbb{G}(k, n)| \prod_{i=1}^{d_{k,n}} \frac{|\Omega_k(X_i)|}{|\mathbb{G}(k, n)|}.$$

Note now that applying Theorem 2.2 to the special real Schubert variety

$$\text{Sch}(k, n) = \{\Lambda \in \mathbb{G}(k, n) : \Lambda \cap \Pi \neq \emptyset\}$$

where  $\Pi$  is a fixed  $(n-k-1)$ -dimensional projective subspace of  $\mathbb{RP}^n$  we obtain

$$\begin{aligned} \delta_{k,n} &= \mathbb{E}_{(g_1, \dots, g_{d_{k,n}}) \in O(n+1)^{d_{k,n}}} \#(g_1 \text{Sch}(k, n) \cap \dots \cap g_{d_{k,n}} \text{Sch}(k, n)) \\ &= \alpha(k+1, n-k) |\mathbb{G}(k, n)| \left( \frac{|\text{Sch}(k, n)|}{|\mathbb{G}(k, n)|} \right)^{d_{k,n}}. \end{aligned}$$

This gives an expression for  $\alpha(k+1, n-k)$ , which substituted into (4) gives (4.1).  $\square$

As a consequence we derive the following corollary, which gives a universal upper bound to our random enumerative problem.

**Corollary 4.2.** *If  $X_1, \dots, X_{d_{k,n}}$  are convex hypersurfaces, then*

$$\tau_k(X_1, \dots, X_{d_{k,n}}) \leq \delta_{k,n} \cdot 4^{d_{k,n}}.$$

*Proof.* This follows immediately from (4.1) and (3.4).  $\square$

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